

# TABLEAUX FOR KEY POLYNOMIALS, DEMAZURE CHARACTERS, AND ATOMS

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ABSTRACT. The Schur function indexed by a partition  $\lambda$  with at most  $n$  parts is the sum of the weight monomials for the Young tableaux of shape  $\lambda$ . Let  $\pi$  be an  $n$ -permutation. We give two descriptions of the tableaux that contribute their monomials to the key polynomial indexed by  $\pi$  and  $\lambda$ . (These polynomials are the characters of the Demazure modules for  $GL(n)$ .) The “atom” indexed by  $\pi$  is the sum of weight monomials of the tableaux whose right keys are the “key” tableau for  $\pi$ . Schur functions and key polynomials can be decomposed into sums of atoms. We also describe the tableaux that contribute to an atom, the tableaux that have a left key equal to a given key, and the tableaux that have a left key bounded below by a given key.

## 1. INTRODUCTION

The core of this paper can be read by any mathematician. After technical definitions are given in Section 2, the main definitions and details for the background material mentioned here appear in Section 3 (which is a second introductory section).

We think of “Demazure” (key) polynomials as being “partial Schur functions”: The Schur function  $s_\lambda(x)$  is the sum of weight monomials for the semistandard tableaux of shape  $\lambda$ . Via the notion of “right key”, specification of an  $n$ -permutation  $\pi$  determines a certain subset of those tableaux; the sum of their weight monomials is the Demazure polynomial we denote  $d_\lambda(\pi; x)$ . These polynomials give a filtration for  $s_\lambda(x)$  indexed by the Bruhat order: As  $\pi$  increases, more of the monomials for  $s_\lambda(x)$  are incorporated into  $d_\lambda(\pi; x)$ .

But in the big view it seems best to take the definition of Demazure polynomial to be the result of applying a sequence of divided difference operators corresponding to  $\pi$  to a weight monomial specified by  $\lambda$ : When studying flag varieties, Demazure developed [Dem] this formula to describe certain characters of a Borel subgroup of any semisimple Lie group. By 1990 Lascoux and Schützenberger [LS2] had developed a combinatorial description of these polynomials using the plactic algebra. A central notion in their work was that of the right key of a given semistandard tableau. They proved that  $d_\lambda(\pi; x)$  arises when a tableau is allowed to contribute its monomial if and only if its right key is dominated by the tableau corresponding to  $\pi$ . We quote this result in Theorem 3.1.

One of the authors of this paper gave a simpler method for finding the right key of a tableau [Wil]. Here we use his “scanning” method to present two descriptions of these contributing “Demazure tableaux” which seem to be more direct and more accessible than those available. Our first description of the possible tableau values for a given location depends upon the values of the tableau in the columns “to the east”. Our second description depends upon the values of the tableau in the locations “to the southwest”. These descriptions can be used in backtracking procedures to generate all of the Demazure tableaux for a given  $(\lambda, \pi)$  pair.

Here are some combinatorial descriptions of right keys and Demazure polynomials (specific to Type A): Theorem 4.3 of [LS2], Theorems 1, 2, 5(1)(2)(3), and 6 of [RS1], Section

3 of [RS2], Appendix A.5 of [Ful], Theorems 4.1, 4.2, 4.7 and 4.10 of [Le1], Theorems 3 and 8 of [Ava], Theorem 1.2 and Corollary 5.1 of [Mas], and Chapter 12 of [LB].

Lascoux has also developed several other related notions (mostly with Schützenberger, but also more recently). The sums of the monomials of the tableaux whose right keys are exactly a given key were also considered in [LS2]; there they were also described with actions of operators. Following Mason, we refer to these polynomials as “atoms”. Schur functions and key polynomials can be expressed as sums of atoms, where the sums run over certain permutations according to Bruhat orders. The notion of the “left key” of a tableau was developed in [LS1]. In that paper Lascoux and Schützenberger considered the tableaux whose left key is one specified key and whose right key is another specified key. All of these considerations would lead us (time permitting) to initially consider eight tableaux description problems: (Right or Left key of the tableaux)  $\times$  (is Bounded by or is Equal to a given key)  $\times$  (referring to entries to the East or to the SouthWest). In addition to the (R,B,E) and (R,B,SW) descriptions mentioned above, we also present (R, Eq, E), (L, Eq, SW), and (L, B, SW) descriptions. These five (eight) descriptions can now (could then) be combined in various ways. One can combine our (R, Eq, E) and (L, Eq, SW) descriptions to describe the tableaux of [LS1] mentioned above. To be nonzero, these polynomials should be indexed by intervals in Bruhat orders. Our (R, B, SW) and (L, B, SW) descriptions can be combined in a more practical fashion to describe a generalization of Demazure polynomials that would be indexed by intervals in Bruhat orders. (Demazure introduced Demazure polynomials while studying the desingularization of Schubert varieties. Kazhdan-Lusztig polynomials are indexed by intervals in Bruhat orders and are related to the structure of singularities of Schubert varieties.)

Reiner and Shimozono’s Theorem 25 of [RS1] and Postnikov and Stanley’s Theorem 14.1 of [PS] related Demazure polynomials to flagged Schur functions for certain  $\pi$ . Postnikov and Stanley then remarked that the sets of Gelfand patterns for the flagged Schur functions that arise in this way form convex polytopes. In [PW] we use one of the Demazure tableaux descriptions of this paper to prove that the set of Demazure tableaux for  $(\lambda, \pi)$  forms a convex polytope if and only if  $\pi$  is “ $\lambda$ -312 avoiding”. A by-product is a sharpening of the [Thm. 25, RS1] description of the relationship between Demazure polynomials and flagged Schur functions; this relationship is now stated at the tableau level.

Demazure characters have been widely studied. Why are atoms of interest? These polynomials have arisen as certain specializations of nonsymmetric Macdonald polynomials [Ion] [HHL] [Mas]. Haglund, Haiman, and Loehr referred to atoms as “nonsymmetric Schur functions”. Mason’s combinatorial description of atoms here helped lead to our [Wi1]. Lascoux has been studying Demazure, Schubert, Grothendieck, and nonsymmetric Macdonald polynomials from the viewpoint of divided difference operators. When doing experiments in this context, one must express the empirical results in terms of the polynomials in some basis. Here he has found [personal communication] atoms to form a particularly useful basis for all polynomials that generalizes the basis of Schur functions for symmetric polynomials. Our Procedure 7.1 gives a ready-to-code backtracking procedure for generating the tableaux for an atom.

Although the polynomials have provided the motivation, our results are set entirely within the finer context of tableaux. The notions of right and left keys were reduced to two scanning descriptions in [Wi1]. So the core of this paper is concerned only with comparing the tableau output of a scanning method to a given key tableau.

In Section 4 we present the scanning method for finding the right key. In Sections 5-9 we state and prove our descriptions of sets of tableaux that are constrained by given keys using our “insider” language of scanning tableaux. In Section 10 we summarize our results for “outsiders” in terms of left and right keys and polynomials. The optional appendix supplies the details needed to view Demazure characters in the context of [Hum].

## 2. BASIC DEFINITIONS AND NOTATION

Let  $p, q \in \mathbb{Z}$ . Set  $[p, q] := \{p, p+1, \dots, q\}$ . Throughout the paper some  $n \geq 1$  is fixed. Set  $[n] := [1, n]$  and  $(n) = (1, 2, \dots, n)$ .

An  $n$ -partition  $\lambda$  is an  $n$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  of integers with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ . Let  $\Lambda_n^+$  denote the set of all  $n$ -partitions. An  $n$ -permutation  $\pi$  is an  $n$ -tuple  $(\pi_1, \pi_2, \dots, \pi_n)$  with distinct entries from  $[n]$ . Let  $S_n$  denote the set of all  $n$ -permutations.

Given some  $\pi \in S_n$ , for  $1 \leq i \leq n-1$  define  $s_i \cdot \pi$  to be the  $n$ -tuple formed from  $\pi$  by interchanging the entries  $i$  and  $i+1$ . Fix  $\pi \in S_n$ . Suppose the composition  $s_{i_t} \dots s_{i_2} s_{i_1}$  is such that  $s_{i_t} \dots s_{i_2} s_{i_1} \cdot (n) = \pi$  with  $t$  minimal. We say  $s_{i_t} \dots s_{i_2} s_{i_1}$  is *reduced* for  $\pi$ . Let  $\pi_0$  denote the permutation  $(n, n-1, \dots, 2, 1)$ .

Let  $x_1, \dots, x_n$  be variables. Let  $P(x)$  be a polynomial in  $x_1, \dots, x_n$ . Re-use the symbols  $s_i$  for  $1 \leq i \leq n-1$  and define  $s_i \cdot P(x)$  to be the polynomial obtained by interchanging  $x_i$  and  $x_{i+1}$  in  $P(x)$ . For  $1 \leq i \leq n-1$  also define operators  $\rho_i := (x_i - x_{i+1})^{-1} \circ (1 - s_i) \circ x_i$  (multiply, swap, subtract, then divide) and  $\bar{\rho}_i = \rho_i - 1$ . Within a monomial  $x_1^{b_1} \dots x_i^{b_i} x_{i+1}^{b_{i+1}} \dots x_n^{b_n}$ , if  $b_i \geq b_{i+1}$  then the “local symmetrizing” operator  $\rho_i$  replaces the  $x_i^{b_i} x_{i+1}^{b_{i+1}}$  factors with the “locally symmetric string” that “connects”  $x_i^{b_i} x_{i+1}^{b_{i+1}}$  to  $x_i^{b_{i+1}} x_{i+1}^{b_i}$ . For example, suppose  $n = 4$ . Using unsubscripted variable names such as  $x := x_2$  for readability, we have  $\rho_2 \cdot w^3 x^7 y^4 z^9 = [\frac{1-s_2}{x-y} x] \cdot w^3 x^7 y^4 z^9 = w^3 (x^7 y^4 + x^6 y^5 + x^5 y^6 + x^4 y^7) z^9$ . The operator  $\bar{\rho}_i$  omits the first term.

Fix  $\lambda \in \Lambda_n^+$ . The *Young diagram* (or *shape*) of  $\lambda$ , also denoted  $\lambda$ , consists of  $\lambda_i$  left justified boxes in the  $i$ th row for  $1 \leq i \leq n$ . Set  $|\lambda| := \lambda_1 + \lambda_2 + \dots + \lambda_n$ . To emphasize the importance of columns over rows, the box in the  $j$ th column and the  $i$ th row is denoted  $(j, i) \in \lambda$ . As in [Wi1], the column lengths of  $\lambda$  are denoted  $\zeta_1, \zeta_2, \dots, \zeta_{\lambda_1}$ . A *semistandard tableau*  $T$  of shape  $\lambda$  is a filling of  $\lambda$  with elements of  $[n]$  such that its values  $T(j, i)$  satisfy  $T(j, i) \leq T(j+1, i)$  and  $T(j, i) < T(j, i+1)$  whenever the subscripts make sense: Use the value  $k$  when  $T(l-1, k)$  is referenced with  $l=1$ , use the value  $n$  when  $T(l+1, k)$  is referenced with  $l=\lambda_k$ , and use the value  $n+1$  when  $T(l, k+1)$  is referenced with  $k=\zeta_l$ . Let  $\mathcal{T}_\lambda$  denote the set of all semistandard tableau of shape  $\lambda$ . For  $T, U \in \mathcal{T}_\lambda$ , we write  $T \leq U$  if  $T(j, i) \leq U(j, i)$  for all  $(j, i) \in \lambda$ ; here we say  $T$  is *dominated by*  $U$ . For  $T \in \mathcal{T}_\lambda$ , let  $m(T)$  denote the maximum of the values that appear at the bottoms of the columns of  $T$ . For the empty tableau  $(())$ , define  $m((())) := 1$ . Given  $T \in \mathcal{T}_\lambda$ , its *weight monomial* is  $x^T := \prod_{i=1}^n x_i^{c_i}$ , where  $c_i$  is the number of values in  $T$  equal to  $i$ . A tableau  $T \in \mathcal{T}_\lambda$  is a *key* if the values in a column also appear in every column to the west of that column. Given  $\pi \in S_n$ , the  $\lambda$ -key of  $\pi$  is the semistandard tableau  $Y_\lambda(\pi)$  of shape  $\lambda$  whose  $j$ th column is obtained by sorting  $\pi_1, \pi_2, \dots, \pi_{\zeta_j}$  into ascending order and then entering these values from top to bottom. The key  $Y_\lambda(\pi_0)$  is the unique maximal element of  $\mathcal{T}_\lambda$ .

The remarks in the next section regarding the decomposition of Schur functions require a more nuanced choice of  $\pi$  relative to  $\lambda$ : Fix some  $\lambda \in \Lambda_n^+$ . Let  $q_1 < q_2 < \dots < q_k$  for some  $k \geq 0$  denote the distinct columns lengths of  $\lambda$ . Set  $Q_\lambda := \{q_1, \dots, q_k\} = \{\zeta_1, \dots, \zeta_{\lambda_1}\}$ .

Set  $q_0 := 0$  and  $q_{k+1} := n$ . Let  $S_n^\lambda$  denote the set of all  $n$ -permutations  $\pi$  such that whenever  $i, j \in [q_{r-1} + 1, q_r]$  with  $i < j$  for some  $1 \leq r \leq k + 1$ , then  $\pi_i < \pi_j$ . Note that  $|S_n^\lambda| = \frac{n!}{q_1!(q_2 - q_1)! \cdots (n - q_k)!}$ . One has  $Q_\lambda \supseteq [n - 1]$  if and only if the parts of  $\lambda$  are distinct. Hence  $S_n^\lambda = S_n$  if and only if  $\lambda_1 > \lambda_2 > \dots > \lambda_n \geq 0$ . The formation of the keys of shape  $\lambda$  of the elements of  $S_n^\lambda$  defines a bijection to the set of all keys of shape  $\lambda$ . The dominance ordering of these keys describes the Bruhat ordering of the elements of the “quotient”  $W^J$  manifestation of  $S_n^\lambda$  described in the Appendix.

### 3. CITED RESULTS; DEMAZURE POLYNOMIAL AND TABLEAU DEFINITIONS

Fix  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ . Let  $s_{i_1} \dots s_{i_2} s_{i_1}$  be reduced for  $\pi$ . The operators  $s_i$  satisfy  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  for  $1 \leq i \leq n - 1$ , and these relations can be used to relate any two reduced compositions for  $\pi$ . We define the *Demazure polynomial*  $d_\lambda(\pi; x) := \rho_{i_1} \dots \rho_{i_2} \rho_{i_1} \cdot x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ . Since the analogous relations  $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$  also hold, these polynomials are well-defined functions of  $\pi$  (and  $\lambda$ ).

For a tableau  $T \in \mathcal{T}_\lambda$ , the *right key*  $R(T)$  is a certain key in  $\mathcal{T}_\lambda$  that can be defined using a jeu de taquin process, as in [App. A.5, Ful]. The following result of [LS2] appeared as Theorem 1 in [RS1]:

**Theorem 3.1.** *The Demazure polynomial  $d_\lambda(\pi; x)$  is the sum of the weight monomials  $x^T$  for  $T \in \mathcal{T}_\lambda$  such that  $R(T) \leq Y_\lambda(\pi)$ .*

Hence we say that  $T \in \mathcal{T}_\lambda$  is a *Demazure tableau* for  $\pi$  if  $R(T) \leq Y_\lambda(\pi)$ . Let  $\mathcal{D}_\lambda(\pi)$  denote the set of such tableaux. With respect to  $s_\lambda(x) = \sum_{T \in \mathcal{T}_\lambda} x^T$ , one can view a Demazure polynomial as a “partial Schur function”. Since  $Y_\lambda(\pi_0)$  is the unique maximal element of  $\mathcal{T}_\lambda$ , we have  $R(T) \leq Y_\lambda(\pi_0)$  for all  $T \in \mathcal{T}_\lambda$ . Thus  $s_\lambda(x)$  is the Demazure polynomial  $d_\lambda(\pi_0; x)$ .

The condition that a tableau  $T$  of shape  $\lambda$  be semistandard can be expressed by a membership criteria  $T(l, k) \in \mathcal{Z}_\lambda(l, k)$  for all  $(l, k) \in \lambda$ , where  $\mathcal{Z}_\lambda(l, k)$  is a set such as  $[T(l - 1, k), T(l, k + 1)]$  or  $[k, \min\{T(l, k + 1) - 1, T(l + 1, k)\}]$ . These could be called the “southwestern” and “southeastern” condition sets. The four obvious ways to construct one semistandard tableau of shape  $\lambda$  by hand entail backtracking with respect to one of the four “quadrant” directions. Our two descriptions of Demazure tableaux, Theorems 5.1 and 8.1, are roughly of this nature.

We define the *atom*  $c_\lambda(\pi; x) := \bar{\rho}_{i_1} \dots \bar{\rho}_{i_2} \bar{\rho}_{i_1} \cdot x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ . This notion is well-defined by similar reasoning. The following result is a consequence of Theorem 3.8 of [LS2]:

**Theorem 3.2.** *The atom  $c_\lambda(\pi; x)$  is the sum of the weight monomials  $x^T$  for  $T \in \mathcal{T}_\lambda$  such that  $R(T) = Y_\lambda(\pi)$ .*

We say that  $T \in \mathcal{T}_\lambda$  is an *exact Demazure tableau* for  $\pi$  if  $R(T) = Y_\lambda(\pi)$ . Let  $\mathcal{C}_\lambda(\pi)$  denote the set of such tableaux.

The set  $\mathcal{D}_\lambda(\pi)$  is the union of the sets  $\mathcal{C}_\lambda(\pi')$  such that  $Y_\lambda(\pi') \leq Y_\lambda(\pi)$ . For the polynomial statement, one must avoid the redundancy in the summation that would arise when  $\lambda$  is not strict. One has  $d_\lambda(\pi; x) = \sum c_\lambda(\pi'; x)$ , where the sum is over all  $\pi' \in S_n^\lambda$  such that  $Y_\lambda(\pi') \leq Y_\lambda(\pi)$ . In particular  $s_\lambda(x) = \sum c_\lambda(\pi'; x)$ , where the sum is over all  $\pi' \in S_n^\lambda$ .

Requiring  $\pi \in S_n^\lambda$  provides a non-redundant indexing of the Demazure polynomials  $d_\lambda(\pi; x)$ . There is a bijective correspondence from such pairs  $(\lambda, \pi)$  to the set  $\mathbb{N}^n$ , where  $\mathbb{N} := \{0, 1, 2, \dots\}$ . Both the sets  $\{d_\lambda(\pi; x)\}$  and  $\{c_\lambda(\pi; x)\}$  for such  $(\lambda, \pi)$  are integral bases

[Corollary 7, RS1] for the polynomial ring  $\mathbb{Z}[x_1, \dots, x_n]$ . When  $\lambda = (1)$  the atoms are  $x_1, x_2, \dots, x_n$ .

The first paper in this series gave a “scanning method” for computing the right key  $R(T)$  of a tableau  $T$ . This method is described in the next section; its output is denoted  $S(T)$ . Here is Theorem 4.5 of [Wi1]:

**Theorem 3.3.** *Let  $T \in \mathcal{T}_\lambda$ . Then  $R(T) = S(T)$ .*

This view of  $R(T)$  made the following known result readily apparent:

**Corollary 3.4.** *Let  $T \in \mathcal{T}_\lambda$ . Then  $T \leq R(T) = S(T)$ .*

Continue to fix  $\lambda \in \Lambda_n^+$  and let  $T \in \mathcal{T}_\lambda$ . The *left key*  $L(T)$  of  $T$  is a certain key in  $\mathcal{T}_\lambda$  that is defined and may be found in manners analogous to those for the right key [Ful] [Wi1]. The scanning description in [Wi1] easily confirms that  $L(T) \leq T$ .

Reiner and Shimozono referred to the polynomials  $d_\lambda(\pi; x)$  as the “key polynomials”  $\kappa_\alpha(x)$  for “compositions”  $\alpha \in \mathbb{N}^n$ . Our definition of the  $d_\lambda(\pi; x)$  largely follows their definition of the  $\kappa_\alpha(x)$ . Their Theorem 1 can be obtained from the second identity stated in Theorem 4.3 of [LS2] by extracting the terms of degree  $|\lambda|$  and projecting the resulting identity to polynomials in  $n$  commuting variables. In [LS2], the element of the free algebra that projected to  $c_\lambda(\pi; x)$  was called a “standard basis”. Our development here reverses the roles of “definition” and “theorem” for standard bases played by Definition 3.7 and Theorem 3.8 of [LS2].

#### 4. THE SCANNING TABLEAU $S(T)$

Fix  $\lambda \in \Lambda_n^+$  and a tableau  $T \in \mathcal{T}_\lambda$ . Here we recall the scanning method of [Wi1] for constructing the scanning tableau  $S(T)$  of  $T$ .

Let  $1 \leq j \leq \lambda_1$ . Working from the bottom of the  $j^{\text{th}}$  column of  $T$  upwards, we construct a “northeasterly” *scanning path* in the shape  $\lambda$  for each box in that column. Begin with the  $i = \zeta_j$  bottommost box case: Initialize the path by  $P(T; j, \zeta_j) := ((j, \zeta_j))$ , its originating box. Scan the column bottoms  $T(h, \zeta_h)$  for  $h > j$  for the earliest  $h$  such that  $T(j, \zeta_j) \leq T(h, \zeta_h)$ . If such an  $h$  exists, append  $(h, \zeta_h)$  to the list  $P(T; j, \zeta_j)$ ; i.e. now we have  $P(T; j, \zeta_j) = ((j, \zeta_j), (h, \zeta_h))$ . Repeat this appending process for the bottom values of  $T$  further to the right of  $(h, \zeta_h)$ , comparing them to the value in the most recently appended location until there does not exist a further weakly larger bottom value. This completes the construction of  $P(T; j, \zeta_j)$ . The values of  $T$  at the locations in  $P(T; j, \zeta_j)$  form the *earliest weakly increasing subsequence (EWIS)* for the sequence of column bottoms weakly to the right of  $(j, \zeta_j)$ . Begin to create  $S(T)$  by defining the value  $S(T; j, \zeta_j)$  to be the last value in this EWIS.

Form the *remnant* tableau  $T^{(j; \zeta_j - 1)}$  by removing the boxes in  $P(T; j, \zeta_j)$  and their values from  $T$ . Since  $T^{(j; \zeta_j - 1)}$  is semistandard, we may to apply  $S(\cdot)$  to it. As  $i$  decrements from  $\zeta_j - 1$  to 1, continue to perform this process using the column bottoms of the diminishing  $T^{(j; i)}$  to produce the other  $\zeta_j - 1$  scanning paths that originate in the  $j^{\text{th}}$  column. For such  $i$ , the path constructed from the column bottoms of  $T^{(j; i)}$  is denoted  $P(T; j, i)$ . Apply this process to all columns of  $T$  to obtain the scanning values  $S(T; j, i)$  for every  $(j, i) \in \lambda$ . Set  $T^{(j; \zeta_j)} := T$ .

**Lemma 4.1.** *Let  $\lambda \in \Lambda_n^+$ . Let  $T \in \mathcal{T}_\lambda$  and  $(j, i) \in \lambda$ . Let  $T'$  be the tableau obtained by removing the first  $(j - 1)$  columns from  $T^{(j; i)}$ . Then  $S(T; j, i) = m(T')$ .*

*Proof.* Once the earlier scanning paths that originate in the  $j^{th}$  column have been removed, the largest of the remaining values that appear at the bottoms of the  $j^{th}$  through  $\lambda_1^{th}$  columns will be the last value appearing in the EWIS that begins at  $(j, i)$ .  $\square$

## 5. RIGHT KEY DOMINATED BY A GIVEN KEY (FROM THE EAST)

For the rest of this paper we assume that some  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$  has been fixed, and that the  $\lambda$ -key  $Y_\lambda(\pi) =: Y$  has been formed. Here we give necessary and sufficient conditions on the values in a tableau  $T$  of shape  $\lambda$  so that its scanning tableau  $S(T)$  is dominated by  $Y_\lambda(\pi)$ .

Fix some  $(l, k) \in \lambda$ . We define a set that contains the “allowable” values for  $T$  at the location  $(l, k)$ . First form a tableau  $U$  from the remnant tableau  $T^{(l;k)}$  by removing the  $1^{st}$  through  $l^{th}$  columns of  $T^{(l;k)}$ . If  $m(U) > Y(l, k)$ , define the set  $A_\lambda(T, \pi; l, k) := \emptyset$ . If  $m(U) \leq Y(l, k)$ , define  $A_\lambda(T, \pi; l, k) := [k, \min\{Y(l, k), T(l, k+1) - 1, T(l+1, k)\}]$ .

**Theorem 5.1.** *Given  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ , let  $T \in \mathcal{T}_\lambda$ . Then  $S(T) \leq Y_\lambda(\pi)$  if and only if  $T(l, k) \in A_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

In words, to construct a Demazure tableau for  $(\lambda, \pi)$ : Suppose that the columns to the east and the boxes to the south of the box at hand in its column have been filled in with “good-so-far” values. Find and remove the scanning paths originating from those boxes to the south. If any of the column bottoms to the east in the remnant tableaux exceed the value of the  $\lambda$ -key for  $\pi$  in the box at hand, then give up. Otherwise one is free to choose any of the “usual” trial values for the box at hand, provided that one does not exceed the given key value there.

*Proof.* Write  $S(T) =: S$ . Let  $(l, k) \in \lambda$ . Since  $T$  is semistandard we have  $k \leq T(l, k) \leq \min\{T(l, k+1) - 1, T(l+1, k)\}$ . By Corollary 3.4 we have  $T(l, k) \leq S(l, k)$ . As above, form  $U$ . By Lemma 4.1 we have  $S(l, k) = \max\{T(l, k), m(U)\}$ .

First suppose that  $S \leq Y$  for  $T$ . So  $T(l, k) \leq Y(l, k)$ . And since  $\max\{T(l, k), m(U)\} \leq Y(l, k)$ , the set  $A_\lambda(T, \pi; l, k)$  is non-empty. Hence  $T(l, k) \in A_\lambda(T, \pi; l, k)$ .

Next suppose that  $T(j, i) \in A_\lambda(T, \pi; j, i)$  for all  $(j, i) \in \lambda$ . Since  $A_\lambda(T, \pi; l, k)$  is non-empty, we have  $m(U) \leq Y(l, k)$ . Also we have  $T(l, k) \leq Y(l, k)$ . Hence  $S(l, k) \leq Y(l, k)$ .  $\square$

## 6. RIGHT KEY EQUAL TO A GIVEN KEY

Here we give necessary and sufficient conditions on the values in  $T$  so that its scanning tableau  $S(T)$  is equal to  $Y_\lambda(\pi)$ .

Fix some  $(l, k) \in \lambda$ . We now define a set  $C_\lambda(T, \pi; l, k)$  that contains the “allowable” values for  $T$  at the location  $(l, k)$ : If  $l = \lambda_1$ , then set  $C_\lambda(T, \pi; l, k) := \{Y(l, k)\}$  for  $1 \leq k \leq \zeta_{\lambda_1}$ . Suppose  $\lambda_1 > l \geq 1$ . Form  $U$  from  $T^{(l;k)}$  as in Section 5. If  $m(U) > Y(l, k)$ , set  $C_\lambda(T, \pi; l, k) := \emptyset$ . If  $m(U) = Y(l, k)$ , set  $C_\lambda(T, \pi; l, k) := [k, \min\{Y(l, k), T(l, k+1) - 1, T(l+1, k)\}]$ . If  $m(U) < Y(l, k)$ , set  $C_\lambda(T, \pi; l, k) := \{Y(l, k)\} \cap [k, \min\{T(l, k+1) - 1, T(l+1, k)\}]$ . Examples of such sets are given in the next section.

**Theorem 6.1.** *Given  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ , let  $T \in \mathcal{T}_\lambda$ . Then  $S(T) = Y_\lambda(\pi)$  if and only if  $T(l, k) \in C_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

*Proof.* The beginning of this proof is the same as the first paragraph of the proof of Theorem 5.1.

First suppose that  $S = Y$  for  $T$ . So  $T(l, k) \leq Y(l, k)$ . Since  $T$  is semistandard we have  $k \leq T(l, k) \leq \min\{T(l, k+1)-1, T(l+1, k)\}$ . Here we have  $\max\{T(l, k), m(U)\} = Y(l, k)$ , and hence  $m(U) \leq Y(l, k)$ . If  $m(U) < Y(l, k)$ , then we must have  $T(l, k) = Y(l, k)$  in order to have  $S(l, k) = Y(l, k)$ . So here  $T(l, k) \in C_\lambda(T, \pi; l, k)$ . If  $m(U) = Y(l, k)$ , one also has  $T(l, k) \in C_\lambda(T, \pi; l, k)$ .

Next suppose that  $T(j, i) \in C_\lambda(T, \pi; j, i)$  for all  $(j, i) \in \lambda$ . Since  $C_\lambda(T, \pi; l, k)$  is non-empty, we have  $m(U) = Y(l, k)$  or  $m(U) < Y(l, k)$ . In the former case, having  $T(l, k) \leq Y(l, k)$  implies that  $S(l, k) = Y(l, k)$ . In the latter case, the definition of  $C_\lambda(T, \pi; j, i)$  implies  $T(l, k) = Y(l, k)$ . So  $Y(l, k) \leq S(l, k) = \max\{T(l, k), m(U)\}$ . Now  $Y(l, k) < \max\{T(l, k), m(U)\}$  would imply  $Y(l, k) < m(U)$ , which is impossible here. Hence  $Y(l, k) = S(l, k)$ .  $\square$

## 7. GENERATION OF TABLEAUX FOR AN ATOM

We continue to work in the context established in Section 6. Here we present a backtracking procedure which generates all tableaux  $T$  of shape  $\lambda$  that have their scanning tableau  $S(T)$  equal to the  $\lambda$ -key  $Y_\lambda(\pi)$ . This procedure constructs each of the desired tableaux from east to west. (The generation procedure on p. 281 of [Le1] produces all of  $\mathcal{D}_\lambda(\pi)$ .)

For  $\lambda_1 \geq j \geq 1$ , denote the partition with column lengths  $\zeta_j, \zeta_{j+1}, \dots, \zeta_{\lambda_1}$  by  $\lambda^{[j]}$ . In the description below, lower portions of the pending new column are denoted by  $L$  and the empty pending column is denoted  $()$ . Each pending new column  $L$  (that will be extended upward) needs to be accompanied by an updated (shrinking upwards) partial tableau  $U$ . The sets of potential new values are denoted by  $C$ . The columns of the growing tableau  $T$  and of the shrinking tableau  $U$  are indexed from the right by  $\lambda_1, \lambda_1 - 1, \lambda_1 - 2, \dots$

**Procedure 7.1.** *Input  $\lambda \in \Lambda_n^+$  and  $\pi \in S_n$ . Let  $\mathcal{V}^{[\lambda_1]}$  be the set consisting of the one tableau  $T$  of shape  $\lambda^{[\lambda_1]}$  that is formed by taking the last column of  $Y_\lambda(\pi) =: Y$ . As  $j$  decrements from  $\lambda_1 - 1$  to 1, successively form sets  $\mathcal{V}^{[j]}$  of tableaux of shapes  $\lambda^{[j]}$  as follows: For each  $T \in \mathcal{V}^{[j+1]}$ , do:*

*Let  $\mathcal{F}_{\zeta_j+1}$  be the set consisting of the one ordered pair  $((), T)$ .*

*As  $i$  decrements from  $\zeta_j$  to 1, successively build up sets  $\mathcal{F}_i$  of ordered pairs as follows:*

*For each  $(L, U) \in \mathcal{F}_{i+1}$ , do:*

*When  $i < \zeta_j$ , let  $t$  be the first (northernmost) value in  $L$ ; when  $i = \zeta_j$ , let  $t$  be  $n + 1$ .*

*If  $m(U) > Y(j, i)$ , set  $C := \emptyset$ .*

*If  $m(U) = Y(j, i)$ , set  $C := [i, \min\{Y(j, i), t - 1, T(j + 1, i)\}]$ .*

*If  $m(U) < Y(j, i)$ , set  $C := \{Y(j, i)\} \cap [i, \min\{t - 1, T(j + 1, i)\}]$ .*

*If  $C$  is empty, then discard  $(L, U)$ .*

*Let  $\mathcal{F}(L, U)$  be the set of all ordered pairs  $(L', U')$  that can be formed by prepending an element  $z$  of  $C$  to  $L$  and then forming  $U'$  by deleting from  $U$  the values and the boxes that lie in the scanning path in  $U$  that originates from the value  $z$  at the location  $(j, i)$ . Let  $\mathcal{F}_i$  be the union of the  $\mathcal{F}(L, U)$  as  $(L, U)$  runs through  $\mathcal{F}_{i+1}$ . If  $\mathcal{F}_i$  is empty, then discard  $T$ . (When  $i = 1$ , each  $U'$  will be the null tableau  $(())$  on the empty shape.)*

*After  $i = 1$ , form the elements of  $\mathcal{V}^{[j]}$  descended from this  $T$  by prepending each column  $L$  that appears in a pair  $(L, (()))$  in  $\mathcal{F}_1$  to the tableau  $T$ . Continue to the next  $T \in \mathcal{V}^{[j+1]}$ .*

*After  $j = 1$ , output the set of semistandard tableaux  $\mathcal{V}^{[1]}$ .*

**Example.** Suppose  $n = 9$  and that  $\lambda$  is such that  $\zeta = (7, 5, 4, 2)$ . Let  $\pi = (6, 8, 3, 7, 4, 1, 9, 2, 5)$ . The first tableau displayed in Figure 1 is  $Y_\lambda(\pi)$ . Within the Procedure 7.1, suppose that we are now at  $j = 2$  and  $i = 3$  and that the values show in the second diagram have been chosen so far. Third in Figure 1 is the partial tableau  $U$  obtained by removing the EWIS's  $(7, 8, 8)$  and  $(6, 7)$  that begin in  $L$  for this partial tableau.

|   |   |   |   |
|---|---|---|---|
| 1 | 3 | 3 | 6 |
| 3 | 4 | 6 | 8 |
| 4 | 6 | 7 |   |
| 6 | 7 | 8 |   |
| 7 | 8 |   |   |
| 8 |   |   |   |
| 9 |   |   |   |

|  |   |   |   |
|--|---|---|---|
|  |   | 3 | 6 |
|  |   | 4 | 8 |
|  |   | 7 |   |
|  | 6 | 8 |   |
|  | 7 |   |   |
|  |   |   |   |
|  |   |   |   |

|   |   |
|---|---|
| 3 | 6 |
| 4 |   |

|   |   |   |   |
|---|---|---|---|
| 1 | 1 | 3 | 6 |
| 2 | 3 | 4 | 8 |
| 4 | 5 | 7 |   |
| 5 | 6 | 8 |   |
| 6 | 7 |   |   |
| 7 |   |   |   |
| 9 |   |   |   |

FIGURE 1

Here  $m(U) = 6 = Y(2, 3)$ , so  $C = [3, \min\{6, 6 - 1, 7\}] = [3, 5] = \{3, 4, 5\}$ . In Figure 2 the respective cases for these three potential values are indexed with the subscripts  $a, b, c$ . Here  $\mathcal{F}(L, U)$  consists of the following 3 ordered pairs:

$$\mathcal{F}(L, U) = \{ (L'_a, U'_a) = \left( \begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline \end{array} \right), (L'_b, U'_b) = \left( \begin{array}{|c|} \hline 4 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline \end{array} \right), (L'_c, U'_c) = \left( \begin{array}{|c|} \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \right) \}.$$

FIGURE 2

For  $(L'_a, U'_a)$ , we have  $m(U'_a) = 3 < 4 = Y(2, 2)$ . Thus  $C = \{4\} \cap [2, \min\{2, 4\}] = \emptyset$ , and so  $(L'_a, U'_a)$  should be discarded. The same applies to  $(L'_b, U'_b)$ . However, for  $(L'_c, U'_c)$ , we have  $C = [2, \min\{4, 4, 4\}] = [2, 4]$ . Hence this process can be continued. In fact, this partial tableau can be filled entirely to produce a tableau whose scanning tableau is  $Y_\lambda(\pi)$ . The last tableau displayed in Figure 1 is one such tableau. Once the proof of Theorem 6.1 is understood, it should be clear that this procedure does indeed generate all of the Demazure tableaux at  $\pi$ .

**Theorem 7.2.** *Let  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ . Then  $\mathcal{V}^{[1]} = \{T \in \mathcal{T}_\lambda \mid S(T) = Y_\lambda(\pi)\}$ .*

## 8. RIGHT KEY DOMINATED BY A GIVEN KEY (FROM THE SOUTHWEST)

Here we show the scanning tableau of a given  $T$  is dominated by the  $\lambda$ -key of  $\pi$  if and only if the values of  $T$  come from a “southwest” condition set.

Fix  $T \in \mathcal{T}_\lambda$ . Fix  $(l, k) \in \lambda$ . For each  $j \leq l$ , it can be seen that there is exactly one  $i \in [1, \zeta_j]$  such that  $(l, k) \in P(T; j, i)$ . Now fix some  $1 \leq j \leq l - 1$ . Let  $a(l, k; j) =: a(j)$  be the row index such that  $(l - 1, k) \in P(T; j, a(j))$ . If  $k < \zeta_l$ , let  $b(l, k; j) =: b(j)$  be the row index such that  $(l, k + 1) \in P(T; j, b(j))$ . When  $k = \zeta_l$ , set  $b(l, k; j) := \zeta_j + 1$ . It can be seen that the only paths beginning in column  $j$  that may reach  $(l, k)$  are the paths originating from rows  $a(j)$  through row  $b(j) - 1$  inclusive.



For  $a(j) \leq i \leq b(j) - 1$ , let  $h$  be the largest value less than  $l$  such that  $(h, m) \in P(T; j, i)$  for some  $m$ . Then for such  $i$ , define  $E(l, k; j, i) := T(h, m)$ , where  $h$  and  $m$  depend upon  $l, k, j, i$  as above. By convention, set  $a(l) := b(l) - 1 := k$  and  $E(l, k; l, k) := k$ . Now refer to  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$  and  $Y_\lambda(\pi) =: Y$ . Define the set  $B_\lambda(T, \pi; l, k) := \bigcap_{j=1}^l \left( \bigcup_{i=a(j)}^{b(j)-1} [E(l, k; j, i), Y_\lambda(\pi; j, i)] \right)$ . The following result appeared in [Wi2]:

**Theorem 8.1.** *Given  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ , let  $T \in \mathcal{T}_\lambda$ . Then  $S(T) \leq Y_\lambda(\pi)$  if and only if  $T(l, k) \in B_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

*Proof.* First fix  $(l, k) \in \lambda$ . Let  $1 \leq j \leq l - 1$ . Let  $1 \leq i \leq \zeta_j$  be the unique index such that  $(l, k) \in P(T; j, i)$ . By hypothesis we have  $S(T; j, i) \leq Y(j, i)$ . The last value before  $T(l, k)$  in the EWIS defining  $P(T; j, i)$  was denoted  $E(l, k; j, i)$ . The last value in this EWIS is  $S(T; j, i)$ . So we have  $E(l, k; j, i) \leq T(l, k) \leq S(T; j, i)$ . Hence  $T(l, k) \in [E(l, k; j, i), Y(j, i)]$ . Note that  $a(j) \leq i \leq b(j) - 1$ . When  $j = l$ , we have  $\bigcup_{h=a(j)}^{b(j)-1} [E(l, k; j, h), Y(j, h)] = [k, Y(l, k)]$ . Since  $T$  is semistandard, we know  $T(l, k) \geq k$ . From the definition of  $S(T; l, k)$ , we have  $T(l, k) \leq S(T; l, k)$ . Hence  $T(l, k) \in [k, Y(l, k)]$ . Intersecting over all  $1 \leq j \leq l$ , we see that  $T(l, k) \in B_\lambda(T, \pi; l, k)$ .

Now fix  $(j, i) \in \lambda$ . Let  $(l, k)$  denote the last position in  $P(T; j, i)$ ; here  $S(T; j, i) = T(l, k)$ . By hypothesis, since  $1 \leq j \leq l$  we have  $T(l, k) \in \bigcup_{h=a(j)}^{b(j)-1} [E(l, k; j, h), Y(j, h)]$ . However, the value  $T(l, k) < E(l, k; j, h)$  for all  $h > i$ . (Otherwise  $(l, k)$  would be in  $P(T; j, h)$  for some  $h > i$ .) So  $T(l, k) \in \bigcup_{h=a(j)}^i [E(l, k; j, h), Y(j, h)]$ . Since  $Y$  is semistandard, we have  $Y(j, r) > Y(j, s)$  when  $r > s$ . Thus  $Y(j, i)$  is an upperbound for  $\bigcup_{h=a(j)}^i [E(l, k; j, h), Y(j, h)]$ . This implies  $S(T; j, i) = T(l, k) \leq Y(j, i)$ .  $\square$

## 9. LEFT KEY CONDITIONS

Here we outline results for the left key of a tableau that are analogous to our Section 5 and 6 right key results. These conditions for a left key to equal or to dominate a given key are expressed in terms of “southwestern” values.

Continue to fix  $\lambda \in \Lambda_n^+$  and now fix  $\sigma \in S_n$ . Form the  $\lambda$ -key  $Y_\lambda(\sigma)$  of  $\sigma$  and let  $T \in \mathcal{T}_\lambda$ .

We denote the *left key* [App. A.5, Ful] of  $T$  by  $L(T)$ . Following Section 5 of [Wi1], we describe the construction of the *left scanning tableau*  $M(T) =: M$ : Let  $1 \leq l \leq \lambda_1$ . Remove the columns to the east of the  $l^{\text{th}}$  column from  $T$  and  $\lambda$ . For an uncluttered presentation we refer to these results also with  $T$  and  $\lambda$ . Consider the value  $T(l, \zeta_l)$  at the bottom of column  $l$  of  $T$ . Successively inspecting the values in the columns indexed by  $l - 1, l - 2, \dots$  find the values that form the *maximizing weakly decreasing sequence (MWDS)* beginning with  $T(l, \zeta_l)$ . To do so, take the maximum value in the next column to the left that is less than or equal to the most recent entry in the sequence. Since  $T$  is semistandard, one value will be taken from each column to the west. The locations of these values form the *left scanning path* originating at  $T(l, \zeta_l)$ . The last value of this path will come from the first column of  $T$ ; it is by definition the value  $M(l, \zeta_l)$ . Remove these values and the values beneath them from  $T$  and the corresponding locations from  $\lambda$ . Repeat this process to successively produce the other values  $M(l, \zeta_l - 1), M(l, \zeta_l - 2), \dots, M(l, 1)$  in the  $l^{\text{th}}$  column of  $M$ . Once this has been done for every  $1 \leq l \leq \lambda_1$ , the left scanning tableau  $M(T)$  has been constructed. According to Section 5 of [Wi1], we have  $L(T) = M(T)$ .

Fix some  $(l, k) \in \lambda$ . First suppose  $l \geq 2$ . Form a tableau  $U$  from  $T$  by first removing the columns of  $T$  and  $\lambda$  to the right of the  $l^{\text{th}}$  column and then successively finding the

left scanning paths originating from locations  $(l, i)$  for  $i = \zeta_l, \zeta_l - 1, \zeta_l - 2, \dots, k + 1$  and successively removing from  $T$  and  $\lambda$  the values and locations that are on or below these paths. Let  $q$  be maximal such that  $U(l - 1, q) \leq T(l, k + 1) - 1$ . Find the left scanning paths in  $U$  that originate at the locations  $(l - 1, q), (l - 1, q - 1), \dots$ . (Do not remove these paths as they are formed.) Note that if  $k \leq h < i \leq q$ , then the path originating at  $(l - 1, h)$  stays weakly above the path originating at  $(l - 1, i)$ . Let  $g_q, g_{q-1}, \dots$  be the ending values in the first column of  $T$  of these paths. Note that the ordering of the paths implies  $g_q \geq g_{q-1} \geq \dots$ . Let  $p$  be minimal such that  $g_p \geq Y_\lambda(\sigma; l, k)$ : The left scanning path originating at  $(l - 1, p)$  is the last path that needs to be considered. If no such  $p$  exists, define the set  $F_\lambda(T, \sigma; l, k) := \emptyset$ . Otherwise define  $F_\lambda(T, \sigma; l, k) := [T(l - 1, p), T(l, k + 1) - 1]$ . When  $l = 1$ , define  $F_\lambda(T, \sigma; l, k) := [Y(\sigma; l, k), T(l, k + 1) - 1]$ .

**Theorem 9.1.** *Given  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ , let  $T \in \mathcal{T}_\lambda$ . Then  $M(T) \geq Y_\lambda(\sigma)$  if and only if  $T(l, k) \in F_\lambda(T, \sigma; l, k)$  for all  $(l, k) \in \lambda$ .*

*Proof.* Let  $T \in \mathcal{T}_\lambda$ . Write  $M(T) =: M$  and  $Y_\lambda(\sigma) =: Y$ . Let  $(l, k) \in \lambda$ . Since the first column of  $M(T)$  is the first column of  $T$ , the case  $l = 1$  is obvious. So suppose  $l \geq 2$ . By semistandardness  $T(l, k) \leq T(l, k + 1) - 1$ . So when forming the MWDS for  $M(l, k)$  we need consider only values within the locations  $(l - 1, q), (l - 1, q - 1), \dots, (l - 1, k)$  where  $q$  is maximal such that  $T(l - 1, q) \leq T(l, k + 1) - 1$  and such that  $(l - 1, q)$  was not in a left scanning path for a location  $(l, h)$  with  $h > k$ . Refer to the definition of  $F_\lambda(T, \sigma; l, k)$  for the entities  $q, p, g_q, g_{q-1}, \dots, g_p$ .

First suppose that  $M \geq Y$  for  $T$ . For the sake of contradiction, suppose  $T(l, k) < T(l - 1, p)$ . Let  $p > h \geq k$  be such that the MWDS from  $(l, k)$  passes through  $(l - 1, h)$ . By the minimality of  $p$  we have  $g_h < Y(l, k)$ . But since  $g_h = M(l, k)$ , this would yield the contradiction  $M(l, k) < Y(l, k)$ . Hence  $T(l, k) \geq T(l - 1, p)$ . So  $T(l, k) \in F_\lambda(T, \sigma; l, k)$ .

Next suppose that  $T(j, i) \in F_\lambda(T, \sigma; j, i)$  for all  $(j, i) \in \lambda$ . Since  $F_\lambda(T, \sigma; l, k)$  is non-empty, we have  $T(l - 1, p) \leq T(l, k) \leq T(l, k + 1) - 1$ . Let  $q \geq h \geq p$  be such that the MWDS from  $(l, k)$  passes through  $(l - 1, h)$ . Then  $M(l, k) = g_h \geq g_p \geq Y(l, k)$ .  $\square$

Now we develop the conditions for having  $M(T) = Y_\lambda(\sigma)$ : Let  $q, p, g_q, \dots, g_p$  be as above. Let  $a$  be minimal and  $b$  maximal such that  $g_a = Y_\lambda(\sigma; l, k) = g_b$ . If no such  $a, b$  exist, define the set  $G_\lambda(T, \sigma; l, k) := \emptyset$ . Otherwise define  $G_\lambda(T, \sigma; l, k) := [T(l - 1, a), \min\{T(l - 1, b + 1) - 1, T(l, k + 1) - 1\}]$ . (When  $l = 1$ , define  $G_\lambda(T, \sigma; l, k) := \{Y(l, k)\}$ .)

**Theorem 9.2.** *Given  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ , let  $T \in \mathcal{T}_\lambda$ . Then  $M(T) = Y_\lambda(\sigma)$  if and only if  $T(l, k) \in G_\lambda(T, \sigma; l, k)$  for all  $(l, k) \in \lambda$ .*

The proof of Theorem 9.2 is similar to that of Theorem 9.1.

## 10. CONCLUSIONS

Using the sets that were developed using the scanning viewpoints in Sections 5-9, the following applications to the original right or left key viewpoint and to polynomials may be stated for a fixed pair of choices  $(\lambda, \pi) \in \Lambda_n^+ \times S_n$ :

**Theorem 10.1.** *Let  $T$  be a semistandard tableau of shape  $\lambda$ . The following are equivalent:*

- (i)  *$T$  is a Demazure tableau for  $\pi$  (that is,  $R(T) \leq Y_\lambda(\pi)$ ),*
- (ii)  *$T(l, k) \in A_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ , and*
- (iii)  *$T(l, k) \in B_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

**Corollary 10.2.** *The Demazure character  $d_\lambda(\pi; x)$  is the sum of  $x^T$  over all  $T \in \mathcal{T}_\lambda$  such that*

- (i)  $T(l, k) \in A_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ , or
- (ii)  $T(l, k) \in B_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .

**Theorem 10.3.** *A semistandard tableau  $T$  of shape  $\lambda$  is a Demazure tableau at exactly  $\pi$  (that is,  $R(T) = Y_\lambda(\pi)$ ) if and only if  $T(l, k) \in C_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

**Corollary 10.4.** *The atom  $c_\lambda(\pi; x)$  is the sum of  $x^T$  over all  $T \in \mathcal{T}_\lambda$  such that  $T(l, k) \in C_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

**Theorem 10.5.** *Procedure 7.1 produces all semistandard tableaux whose right keys are the  $\lambda$ -key of  $\pi$ , that is  $\mathcal{V}^{[1]} = \{T \mid R(T) = Y_\lambda(\pi)\}$ .*

**Corollary 10.6.** *The atom  $c_\lambda(\pi; x)$  is the sum of  $x^T$  over all  $T \in \mathcal{V}^{[1]}$ .*

Now also fix some  $\sigma \in S_n$ :

**Theorem 10.7.** *A semistandard tableau  $T$  has  $Y_\lambda(\sigma) \leq L(T)$  if and only if  $T(l, k) \in F_\lambda(T, \sigma; l, k)$  for all  $(l, k) \in \lambda$  and it has  $Y_\lambda(\sigma) = L(T)$  if and only if  $T(l, k) \in G_\lambda(T, \sigma; l, k)$  for all  $(l, k) \in \lambda$ .*

The polynomial appearing in the Proposition on Slide 15 of [Las] was denoted  $K_\lambda^\mathcal{F} \otimes \hat{K}_\lambda^\mathcal{F}$ .

**Corollary 10.8.** *The polynomial  $K_\lambda^\mathcal{F} \otimes \hat{K}_\lambda^\mathcal{F}$  is the sum of  $x^T$  over all  $T \in \mathcal{T}_\lambda$  such that  $T(l, k) \in G_\lambda(T, \sigma; l, k) \cap C_\lambda(T, \pi; l, k)$  for all  $(l, k) \in \lambda$ .*

Has the polynomial  $\sum x^T$ , sum over all  $T$  such that  $Y_\lambda(\sigma) \leq L(T) \leq T \leq R(T) \leq Y_\lambda(\pi)$ , appeared elsewhere? Here  $T(l, k) \in F_\lambda(T, \sigma; l, k) \cap B_\lambda(T, \pi; l, k)$ , an intersection of two southwestern condition sets. For this polynomial (or  $K_\lambda^\mathcal{F} \otimes \hat{K}_\lambda^\mathcal{F}$ ) to be non-zero, one must have  $\sigma \leq \pi$  in a Bruhat order.

All of our results are “stable” as  $n \rightarrow \infty$  for  $\pi \in S_\infty$  (defined in [RS1]). So the polynomial results hold in infinitely many variables  $x_1, x_2, \dots$  for unbounded tableaux.

#### APPENDIX: INTERFACE WITH REPRESENTATION THEORY

The ingredients needed to define Demazure modules of semisimple Lie algebras and their characters are in [Hum]: Given a complex semisimple Lie algebra  $L$ , choose a Cartan subalgebra  $H$  and a Borel subalgebra  $B \supseteq H$ . These choices determine the rank  $n := \dim(H)$  of  $L$ , and then (for  $1 \leq i \leq n$ ) the simple roots  $\alpha_i \in H^*$ , the simple reflections  $s_i$  of  $H^*$ , and the fundamental weights  $\omega_i$ . The simple reflections generate the Weyl group  $W$  and the fundamental weights generate the weight lattice  $\Lambda$ , which contains the set of dominant weights  $\Lambda^+$ . Fix  $\lambda \in \Lambda^+$ . Let  $V_\lambda$  be a finite dimensional irreducible  $L$ -module with highest weight  $\lambda$ . Let  $w \in W$ . Let  $v_{w\lambda} \neq 0$  be a weight vector of weight  $w\lambda$ . The Demazure module  $D_\lambda(w)$  is the  $B$ -submodule  $\mathcal{U}(B).v_{w\lambda}$  of  $V_\lambda$ , where  $\mathcal{U}(B)$  is the universal enveloping algebra of  $B$ . The lowest weight of this module is  $w\lambda$ . When  $w$  is the longest element  $w_0$  of  $W$ , one has  $D_\lambda(w_0) = V_\lambda$ . For each  $\mu \in \Lambda$  there is a formal exponential  $e^\mu$ . Given  $\mu \in \Lambda$ , let  $m_\lambda(w, \mu)$  be the dimension of the  $H$ -weight space of  $D_\lambda(w)$  of weight  $\mu$ . The formal character  $\text{char}_\lambda(w)$  of  $D_\lambda(w)$  is  $\sum m_\lambda(w, \mu)e^\mu$ , where the sum runs over  $\mu \in \Lambda$  for which  $m_\lambda(w, \mu) \neq 0$ . The formal character of the  $L$ -module  $V_\lambda$  is  $\text{char}_\lambda(w_0)$ . For some  $k \geq 0$ , let  $s_{i_k} \dots s_{i_2} s_{i_1}$  be a reduced decomposition for  $w$ . The Demazure character formula [Kum, Eqn. 8.2.9.4] is  $\text{char}_\lambda(w) = D_{i_k} \dots D_{i_2} D_{i_1}.e^\lambda$ , where

$D_i(e^\mu) := (e^\mu - e^{s_i\mu - \alpha_i})/(1 - e^{-\alpha_i})$  for  $\mu \in \Lambda$ . To precisely index the Demazure submodules of  $V_\lambda$ , first set  $J := J_\lambda := \{i \in [n] : s_i \cdot \lambda = \lambda\}$ . Here  $\text{Stab}_W(\lambda) = \langle s_i : i \in J \rangle =: W_J$ . (If  $\lambda = \sum_{1 \leq i \leq n} a_i \omega_i$  for some  $a_i \in \mathbb{N}$ , then  $J = \{i \in [n] : a_i = 0\}$ .) There is one distinct Demazure submodule for each coset  $wW_J$  in the set of cosets  $W^J := W/W_J$ . Each such coset has a unique minimal length representative in  $W$ ; let  $W^\lambda$  denote the set of these representatives.

Now take  $L$  to be the simple Lie algebra  $sl_n(\mathbb{C})$ ; it has rank  $n - 1$ . Here  $W \cong S_n$ , the symmetric group. Choose  $H$  to be the subspace of diagonal matrices and  $B$  to be the subalgebra of trace free upper triangular matrices. For  $1 \leq i \leq n$ , define  $\phi_i \in H^*$  to be the linear function that extracts the coefficient of the elementary matrix  $E_{ii}$  for each element of  $H$ . Note that  $\phi_1 + \phi_2 + \dots + \phi_n = 0$  on  $H$ . Let  $\mathbf{E}$  denote the real span of  $\phi_1, \phi_2, \dots, \phi_n$ . For  $1 \leq i \leq n - 1$ , we have  $\alpha_i = \phi_i - \phi_{i+1}$  and  $\omega_i = \phi_1 + \phi_2 + \dots + \phi_i$  on  $H$ .

In this paper we avoid using an action from the right (or mentioning  $w^{-1}$ ) by using *two* combinatorial models for the action of  $W$  from the left. For the first model, note that  $s_i \cdot \phi_i = \phi_{i+1}$ ,  $s_i \cdot \phi_{i+1} = \phi_i$ , and  $s_i \cdot \phi_j = \phi_j$  when  $j \notin \{i, i+1\}$ . Set  $x_i := e^{\phi_i}$ . Note that  $x_1 x_2 \dots x_n = 1$ . There is an induced action of  $W$  on the set of formal exponentials: Here  $s_i \cdot x_i = x_{i+1}$ ,  $s_i \cdot x_{i+1} = x_i$ , and  $s_i \cdot x_j = x_j$  when  $j \notin \{i, i+1\}$ . This is the same as the second action of the  $s_i$  in Section 2. Here we say that  $W$  is “acting by value” on the subscripts. This induces the first action of the  $s_i$  in Section 2, on permutations. When using this model for  $W$ , we often refer to the permutation  $\pi := (\pi_1, \dots, \pi_n) := \pi_w := w \cdot (n)$ .

Each  $\mu \in \Lambda$  may be uniquely represented in the form  $\sum_{1 \leq i \leq n-1} b_i \omega_i$  for some  $b_i \in \mathbb{Z}$ . Fix some  $\lambda \in \Lambda^+$  and write  $\lambda =: \sum_{1 \leq i \leq n-1} a_i \omega_i$  for some  $a_i \in \mathbb{N}$ . Here the symbol  $\lambda$  is being used in the traditional Lie-theoretic manner. Transitioning to the traditional combinatorial usage of  $\lambda$ , set  $\lambda_i := \sum_{i \leq j \leq n-1} a_j$  for  $1 \leq i \leq n$  and  $\lambda_n := 0$ . This is the  $i$ th coefficient of  $\lambda$  with respect to the  $\{\phi_i\}$  spanning set for  $\mathbf{E}$  when  $\lambda_n$  is required to vanish. Since  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$ , this produces an  $n$ -partition which will also be denoted  $\lambda$ . This partition  $\lambda$  is strict if and only if the weight  $\lambda$  is strongly dominant. For the second combinatorial model of the action of  $W$ , note that for  $1 \leq i \leq n - 1$  one has the reflection action  $s_i \cdot (\lambda_1, \dots, \lambda_i, \lambda_{i+1}, \dots, \lambda_n)^T = (\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots, \lambda_n)^T$  on column vectors of coefficients with respect to  $\{\phi_i\}$ . Here we say that  $W$  “acts by position”. When using this model for  $W$ , we often depict  $w \in W$  with a reduced decomposition  $s_{i_k} \dots s_{i_2} s_{i_1}$  for some  $k \geq 0$ . The orbit  $W\lambda$  consists of all of the “shuffles” of the multiset of  $n$  integers  $\{\lambda_i\}_{1 \leq i \leq n}$ ; these are called “compositions” in [RS1]. Note that  $J = \{i \in [n - 1] : \lambda_i = \lambda_{i+1}\}$ , and so  $J$  can be used to describe the multiplicities amongst the  $\lambda_i$ . These shuffles correspond exactly to the elements of  $W^\lambda$ . The set of column lengths that may possibly occur in the Young diagram of  $\lambda$  is  $[n - 1]$ . Since  $J$  is the set of “missing” column lengths, comparing to Section 2 we have  $J = [n - 1] - Q_\lambda$ .

The Weyl character formula for the coordinatization of  $\text{char}_\lambda(w_0; x)$  for  $sl_n(\mathbb{C})$  is the bialternant definition of the Schur function  $s_\lambda(x)$ . So  $s_\lambda(x) = \sum_{T \in \mathcal{T}_\lambda} x^T$  implies the dimension  $m_\lambda(w_0, \mu)$  is the number of tableaux  $T$  such that  $c_i$  is the  $i$ th coefficient of  $\mu$  with respect to the  $\{\phi_i\}$  spanning set when the coefficients of  $\mu$  are required to sum to  $|\lambda|$ . Let  $T^+$  be the tableau of shape  $\lambda$  that has  $T^+(j, i) = i$  for  $1 \leq i \leq n$  and  $1 \leq j \leq \lambda_i$ . Here  $x^{T^+}$  is the coordinatization  $x_1^{\lambda_1} \dots x_{n-1}^{\lambda_{n-1}} x_n^0$  of  $e^\lambda$ .

Consider a composition  $\alpha \in W\lambda$ . Let  $w \in W^\lambda$  be of length  $k \geq 0$  such that  $w \cdot \lambda = \alpha$ . Let  $s_{i_k} \dots s_{i_2} s_{i_1}$  be a reduced decomposition for  $w$ , and find  $\pi = \pi_w$ . To relate to the “right action” of [RS1], note that  $\lambda_i = \alpha_{\pi_i}$  for  $1 \leq i \leq n$ . Now identify each value  $1 \leq i \leq n$  of

an  $n$ -semistandard tableau  $T$  of shape  $\lambda$  with the formal exponential  $x_i$ . Define  $w.T^+$  to be the result of replacing each value  $i$  by  $\pi_i$  and resorting the values within each column so that they increase from north to south. Clearly  $w.T^+ = Y_\lambda(\pi)$ . The combinatorial weight  $x^{w.T^+}$  of this tableau is the coordinatization of  $e^{w\lambda}$ . In [RS1] the “key”  $key(\alpha)$  of the composition  $\alpha$  is defined to be the tableau whose first  $\alpha_j$  columns contain the value  $j$  for  $j \geq 1$ . Taking  $j := \pi_i$  for a given  $i \geq 1$ , one sees that the tableau  $w.T^+$  satisfies that definition. Hence  $Y_\lambda(\pi) = key(\alpha)$ .

It is not hard to see that the coordinatization of the Demazure character formula above is our definition (when  $\lambda_n = 0$ ) of the Demazure polynomial  $d_\lambda(\pi; x)$  in Section 2. The dimension  $m_\lambda(w, \mu)$  is the number of Demazure tableaux  $T$  such that  $c_i$  is the  $i^{th}$  coefficient of  $\mu$ .

Since the reductive Lie algebra  $gl_n(\mathbb{C})$  is not semisimple, its Demazure modules are rarely considered in geometric or algebraic papers. However, its coordinatized characters have some aesthetic advantages over those for  $sl_n(\mathbb{C})$ . Since the scalar matrices are in the center of  $gl_n(\mathbb{C})$ , the familiar constructions (such as with tensors or global sections of line bundles on  $SL_n(\mathbb{C})/B$ ) of an  $sl_n(\mathbb{C})$  module  $V_\lambda$  may be readily extended to  $gl_n(\mathbb{C})$ . Once the relation  $\phi_1 + \phi_2 + \dots + \phi_n = 0$  is no longer present, the finite dimensional irreducible polynomial characters of  $gl_n(\mathbb{C})$  are indexed by the Young diagrams for  $n$ -partitions: for each column of length  $n$  in the shape of a  $\lambda \in \Lambda_n^+$ , the formal character has a factor of  $x_1 x_2 \cdots x_n$ . Every Schur function  $s_\lambda(x)$  for  $\lambda \in \Lambda_n^+$  now arises as a formal character for  $gl_n(\mathbb{C})$ . Since the underlying vector spaces for the modules are unchanged, their structure with respect to  $W \cong S_n$  remain the same. Extend the Borel subalgebra  $B$  to the subalgebra  $B'$  of all upper triangular matrices in  $gl_n(\mathbb{C})$ , consider highest weight vectors  $v$  for all  $n$ -partitions  $\lambda \in \Lambda_n^+$ , and construct  $\mathcal{U}(B').wv$  for any  $w \in W$ . Now that the condition  $\lambda_n = 0$  has been removed, *all* of the Demazure polynomials  $d_\lambda(\pi; x)$  considered in this paper arise as the formal characters for such “polynomial” Demazure modules of  $gl_n(\mathbb{C})$  as the initial monomial  $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$  ranges through all  $\lambda \in \Lambda_n^+$ . Columns of length  $n$  in a tableau  $T \in \mathcal{T}_\lambda$  must contain the values  $1, 2, \dots, n$ . It can be seen that such columns are combinatorially inert in this paper. Hence our Theorem 10.1 may be applied to the Demazure characters for  $sl_n(\mathbb{C})$  by requiring  $\lambda$  to be an  $(n-1)$ -partition and invoking the relation  $x_1 x_2 \cdots x_n = 1$ .

In the semisimple  $L$  and coordinatized  $sl_n(\mathbb{C})$  discussions above, to avoid redundant considerations of cases we required  $w \in W^\lambda$ . A permutation  $\pi \in S_n$  corresponds to a  $w \in W^\lambda$  if and only if  $\pi \in S_n^\lambda$ . The criteria for redundancy is the same for the  $gl_n(\mathbb{C})$  case (when  $\lambda$  is any  $n$ -partition) as for the  $sl_n(\mathbb{C})$  case (when  $\lambda_n = 0$ ): whether any of the parts of  $\lambda$  are repeated. Our input permutations  $\pi \in S_n$  are converted to  $\lambda$ -keys  $Y_\lambda(\pi)$ , which are essentially their projections to  $S_n^\lambda$ . For a fixed  $\lambda$ , let  $\pi, \pi' \in S_n^\lambda$  correspond to  $w, w' \in W^\lambda$ . Then  $w' \leq w$  in the Bruhat order on  $W^\lambda$  if and only if  $Y_\lambda(\pi') \leq Y_\lambda(\pi)$  by [Thm. 2.6.3, BB]. This gives the restatement  $d_\lambda(w; x) = \sum c_\lambda(w; x)$ , sum over all  $w' \in W^\lambda$  such that  $w' \leq w$  in the Bruhat order on  $W^\lambda$ .

Several authors have developed the notions of left and right keys for representations of semisimple Lie and Kac-Moody algebras; one recent version is stated within the alcove model of Lenart and Postnikov [Le2].

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